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Lyapunov Stability Analysis of Daisy Chain Control Allocation

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A demonstration that feedback control of systems with redundant controls can be reduced to feedback control of systems without redundant controls and control allocation is presented. It is shown that control allocation can introduce unstable zero dynamics into the system, which is important if input/output inversion control techniques are utilized. The daisy chain control allocation technique for systems with redundant groups of controls is also presented. Sufficient conditions are given to ensure that the daisy chain control allocation does not introduce unstable zero dynamics into the system. Aircraft flight control examples are given to demonstrate the derived results.

I. Introduction

DESIGNING feedback control systems with redundant controls is very prevalent in the aerospace industry today, particularly in the area of flight control.[1–9] Management of the redundant controls is frequently referred to as control allocation.

There recently has been an increase in control allocation research.[10–16] Many control allocation techniques are available to the control engineer and have been proven through extensive research efforts and/or flight programs. Ganging control allocation is presented in a contracted research effort at Honeywell,[5] and Berg et al.[10] use ganging for a simple aircraft example. A relative control effectiveness allocation technique is used in conjunction with pseudocontrols by Lallman.[6,7] The pseudoinverse control allocation technique gives the minimum-norm control solution, and its variations seem to be the most popular control allocation technique in the literature.[5,8,9,16] Lowenberg[15] uses bifurcation analysis to develop a control allocation technique. Bordignon and Durham,[11] Durham,[12,13] and Grogan et al.[14] explore tailored generalized inverses and methods to achieve maximum attainable aircraft moments. Finally, the daisy chain control allocation method[1–5,10] is considered when there exist redundant groups of control effectors.

Interpreting the feedback control of systems with redundant controls as control allocation and feedback control of systems without redundant controls allows practical control command requirements to be addressed separately from some feedback control issues. Minimum control effort, control prioritization, minimum radar cross section, and minimum drag are a few examples of control command requirements. The treatment of control allocation separately, however, can introduce unstable zero dynamics into the system without redundant controls. Thus, if a technique that inverts the plant is used for feedback control, the control allocation could make the closed-loop system unstable. Therefore, care must be taken when designing control allocators independently of the feedback controller.

This paper starts with an explanation of how feedback control of systems with redundant controls is reduced to feedback control of nonredundant systems and control allocation. It is then shown by example that control allocation can introduce unstable zero dynamics into the nonredundant systems. The daisy chain control allocation technique is then presented for the case when there exist redundant groups of controls. A result is given that provides sufficient asymptotic stability conditions for the zero dynamics of systems with daisy chain control allocation. Finally, flight control examples are given that demonstrate the result.

II. Feedback Control of Systems with Redundant Constrained Inputs

Consider the control system in Fig. 1, where $k(\,)$ is a generally nonlinear controller, $f_{\text{act}}(\,)$ is a generally nonlinear actuator model, and G is a linear plant that has the following time-domain representation:

$$\begin{bmatrix} \dot z \\ \dot y \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} B_z \\ B_y \end{bmatrix} u$$

$$z \in \mathbb{R}^{n_z}, \qquad y \in \mathbb{R}^{n_y}, \qquad u \in \mathbb{R}^m \quad (1)$$

It is assumed that the rank of the control effectiveness matrix corresponding to y is equal to the dimension of y, i.e., all of the states in the y partition are independently controllable. This implies that there are at least as many controls as states in the y partition

$$\operatorname{rank}(B_y) = n_y \Rightarrow m \ge n_y \quad (2)$$

When $m > n_y$, the y subsystem is said to have redundant controls because there exists at least one column in B_y that can be removed without reducing its rank. Therefore, we will refer to a redundant system as the system in Eq. (1) with the assumption contained in Eq. (2) and $m > n_y$.

Assume that the actuator model in Fig. 1 consists only of an element by element saturation function

$$f_{\text{act}}(u^{\text{cmd}}) = \text{SAT}(u^{\text{cmd}}) \equiv \begin{bmatrix} \text{sat}_1\left(u_1^{\text{cmd}}\right) & \cdots & \text{sat}_m\left(u_m^{\text{cmd}}\right) \end{bmatrix}^T \quad (3)$$

where $\text{sat}_i(u_i^{\text{cmd}})$ is a standard saturation function of the form in Fig. 2 with upper and lower limits given by $\bar u_i$ and $\underline{u}_i$ respectively. The actuator model of Eq. (3) leads to a constrained control subset[12] defined by the actuator limits

$$\Omega \equiv \{u \mid \underline{u}_i \le u_i \le \bar u_i, i = 1, \ldots, m\} \subset \mathbb{R}^m \quad (4)$$

If $\dot y$ in Eq. (1) is interpreted as the y rate, then $(A_{yz}z + A_{yy}y)$ is the y rate due to the states and $B_y u$ is the y rate due to the controls. It is convenient to define the y rate due to the controls as just the y rate,

$$r_y \equiv B_y u \quad (5)$$

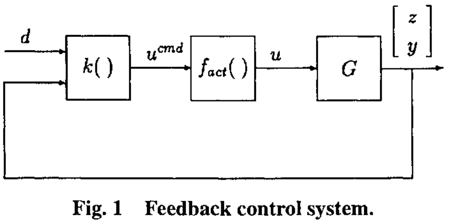

Fig. 1 Feedback control system.

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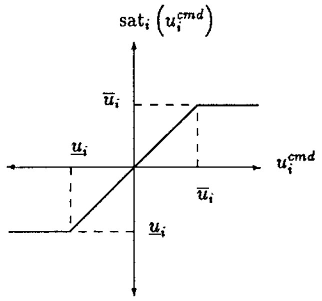

Fig. 2 Standard saturation function.

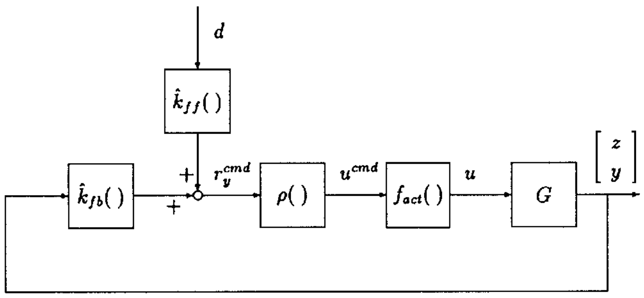

Fig. 3 Feedback control system with control allocation.

Now consider a two-part control law of the following form for the system in Fig. 1:

$$u^{\mathrm{cmd}} = \rho\left(r_y^{\mathrm{cmd}}\right), \qquad r_y^{\mathrm{cmd}} \equiv \hat{k}_{\mathrm{fb}}\left(\begin{bmatrix} z \\ y \end{bmatrix}\right) + \hat{k}_{\mathrm{ff}}(d) \tag{6}$$

where ρ is defined as the control allocation function, r_y^{cmd} is the desired y rate, d is an external disturbance or command, and $\hat{k}_{\mathrm{ff}}$ and $\hat{k}_{\mathrm{fb}}$ are feedforward and feedback compensation, respectively, for the nonredundant system to be defined.

The control law in Eq. (6) results in the control system in Fig. 3. The signal r_y^{cmd} is interpreted as a reduced dimension control command, and ρ maps reduced control commands to actuator commands. The original plant, the actuator model, and the control allocation are grouped together to form a nonredundant nonlinear plant

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} B_z \\ B_y \end{bmatrix}\mathrm{SAT}\left[\rho\left(r_y^{\mathrm{cmd}}\right)\right] \tag{7}$$

If $u^{\mathrm{cmd}} \equiv \rho(r_y^{\mathrm{cmd}}) \in \Omega$, then the nonredundant system becomes

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} B_z\rho\left(r_y^{\mathrm{cmd}}\right) \\ B_y\rho\left(r_y^{\mathrm{cmd}}\right) \end{bmatrix} \tag{8}$$

Thus far there are no restrictions on the control allocation function; however, if the control allocation function is chosen to satisfy the following equation:

$$B_y\rho\left(r_y^{\mathrm{cmd}}\right) = r_y^{\mathrm{cmd}} \tag{9}$$

then the y-rate equation is linear:

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} B_z\rho\left(r_y^{\mathrm{cmd}}\right) \\ r_y^{\mathrm{cmd}} \end{bmatrix} \tag{10}$$

where r_y^{cmd} is interpreted as the reduced or nonredundant control.

The control limits of Eq. (3) that define Ω may be extended to the nonredundant controls by defining an attainable y-rate subset, Φ^y:

$$\Phi^y \equiv \left\{r_y^{\mathrm{cmd}} \mid B_y\rho\left(r_y^{\mathrm{cmd}}\right) = r_y^{\mathrm{cmd}}, u^{\mathrm{cmd}} = \rho\left(r_y^{\mathrm{cmd}}\right) \in \Omega\right\} \subset \mathbb{R}^{n_y} \tag{11}$$

Thus, the system in Eq. (10) is a reasonable open-loop representation of the plant and the control allocation if $r_y^{\mathrm{cmd}} \in \Phi^y$. As seen from Eq. (11), the attainable y-rate subset depends on the control allocation function. Algorithms exist in the literature[12,13] that address the computation of Φ^y. Further, it is shown by Durham[12] that different control allocation functions will result in attainable rate subsets of varying volume.

Now the feedback control problem of finding $u^{\mathrm{cmd}} \in \Omega$ for the redundant system in Eq. (1) has become the feedback control problem of finding $r_y^{\mathrm{cmd}} \in \Phi^y$ for the nonredundant system in Eq. (10) and the control allocation problem of finding a ρ to satisfy Eq. (9).

Stabilization

Consider stabilization of the redundant system in Eq. (1). The control law is assumed to take the form of Eq. (6) with $d = 0$ and $\hat{k}_{\mathrm{ff}}(0) = 0$ because loop stability is not affected by external signals. Dynamic inversion[4] is used to design $\hat{k}_{\mathrm{fb}}$ that stabilizes the nonredundant system in Eq. (10) and is implemented with some control allocation function in the control law of Eq. (6) to stabilize the redundant system of Eq. (1). Consider the following dynamic inversion control law:

$$r_y^{\mathrm{cmd}} = \hat{k}_{\mathrm{fb}}\left(\begin{bmatrix} z \\ y \end{bmatrix}\right) = -A_{yz}z - A_{yy}y + v(y) \tag{12}$$

where $v(y)$ is any additional feedback such that the origin of $\dot{y} = v(y)$ is asymptotically stable. Note that $v(y)$ is possibly dynamic but is not a function of z. Asymptotic stabilizability conditions are given by Byrnes and Isidori[17] for minimum phase systems with full state feedback. Therefore, it is useful to show that the system in Eq. (10) is minimum phase with respect to input r_y^{cmd} and output y. Note that a system is minimum phase if and only if it has asymptotically stable zero dynamics, which are now defined.

The dynamics when $y = 0$ and $\dot{y} = 0$ are referred to as the zero dynamics.[4] It is easily shown that the nonredundant closed loop with the dynamic inversion control law is asymptotically stable if the zero dynamics are asymptotically stable. Therefore, the zero dynamics with respect to input r_y^{cmd} and output y are examined. The input required to maintain $y = 0$ and $\dot{y} = 0$ is found by examining the y dynamics

$$\begin{gathered} \dot{y} = A_{yz}z + A_{yy}y + r_y^{\mathrm{cmd}} \\ y = 0,\ \dot{y} = 0 \Rightarrow r_y^{\mathrm{cmd}} = -A_{yz}z \end{gathered} \tag{13}$$

Inserting $y = 0$ and r_y^{cmd} from Eq. (13) into the z dynamics of Eq. (10) gives the zero dynamics

$$\dot{z} = A_{zz}z + B_z\rho(-A_{yz}z) \tag{14}$$

Therefore, the zero dynamics with respect to input r_y^{cmd} and output y of the system in Eq. (10) depend on the control allocation function ρ. The following example shows that control allocation can introduce unstable zero dynamics into the nonredundant system if control allocation is approached in a naive manner.

Example 1

It is shown that two valid control allocation functions can result in different stability properties of the nonredundant system zero dynamics. Consider the following redundant system of the form in Eq. (1), which is not intended to have any physical meaning:

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -0.06 & A_{zy} \\ 0.51 & A_{yy} \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} -0.4 & 0.3 \\ 1.34 & 1.13 \end{bmatrix}u \tag{15}$$

Now consider the following two different linear control allocation functions:

$$\begin{aligned} \rho_1\left(r_y^{\mathrm{cmd}}\right) &= R_1 r_y^{\mathrm{cmd}} \qquad & R_1 &= [0.525 \quad 0.263]^T \\ \rho_2\left(r_y^{\mathrm{cmd}}\right) &= R_2 r_y^{\mathrm{cmd}} \qquad & R_2 &= [0.436 \quad 0.368]^T \end{aligned} \tag{16}$$

A valid control allocation function must satisfy Eq. (9), and linear functions that satisfy Eq. (9) are just right inverses of B_y. The control

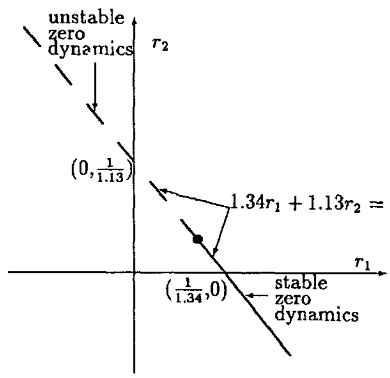

Fig. 4 Zero dynamics stability region.

allocation functions in Eq. (16) satisfy Eq. (9) because they are right inverses of B_y:

$$
\begin{aligned}
B_y R_1 &= [1.34 \quad 1.13] \quad [0.525 \quad 0.263]^T = 1 \\
B_y R_2 &= [1.34 \quad 1.13] \quad [0.436 \quad 0.368]^T = 1
\end{aligned}
\tag{17}
$$

The zero dynamics for the two control allocation functions are given by

$$
\begin{aligned}
\dot{z} &= (A_{zz} - B_z R_1 A_{yz})z = 0.007z \\
\dot{z} &= (A_{zz} - B_z R_2 A_{yz})z = -0.027z
\end{aligned}
\tag{18}
$$

according to Eq. (14). The second control allocation function results in asymptotically stable zero dynamics because $A_{zz} - B_z R_2 A_{yz} < 0$, and the first function results in unstable zero dynamics because $A_{zz} - B_z R_1 A_{yz} > 0$. For this simple example, precise regions of stability are easily seen in Fig. 4. The axes represent elements of the linear right inverse (r_1, r_2), and the diagonal line represents all valid linear control allocation solutions, i.e., right inverses of B_y. The dashed part of the line represents the region of unstable zero dynamics, and the solid part of the line represents the region of stable zero dynamics. The dot represents marginally stable zero dynamics, i.e., $A_{zz} - B_z R A_{yz} = 0$.

The daisy chain control allocation structure for redundant groups of controls is now examined. The following section also develops sufficient conditions to ensure asymptotically stable zero dynamics for the system in Eq. (10) with the daisy chain control allocation.

III. Redundant Groups of Constrained Controls

This section addresses the case when there are only two redundant groups of controls, which are defined subsequently. All of the results are easily extended for an arbitrary number of control groups, but the main ideas are sufficiently conveyed for only two redundant groups of controls. Further, the calculations are easier to follow with only two control groups.

The theorem presented in this section utilizes vector and matrix norms. The Euclidean norm is used for constant real-valued vectors, and the induced matrix 2-norm is used for constant real-valued matrices. The Euclidean norm for vectors and the induced matrix 2-norm for matrices are just the maximum singular value denoted $\bar{\sigma}$:

$$
\|X\| \equiv \bar{\sigma}(X) \;\forall\; X \in \mathbb{R}^{n \times m} \tag{19}
$$

Daisy Chain Control Allocation

Assume the controls are partitioned into two groups defined by the following:

$$
\begin{bmatrix} B_z \\ B_y \end{bmatrix} u = \begin{bmatrix} B_{z_1}\mu_1 + B_{z_2}\mu_2 \\ B_{y_1}\mu_1 + B_{y_2}\mu_2 \end{bmatrix}, \qquad \mu_i \in \mathbb{R}^{m_i}, \qquad i = 1, 2 \tag{20}
$$

If the control effectiveness matrices corresponding to each control group have rank equal to the dimension of y:

$$
\operatorname{rank}\left(B_{y_i}\right) = n_y \Rightarrow m_i \ge n_y \tag{21}
$$

the groups μ_1 and μ_2 are said to be redundant.

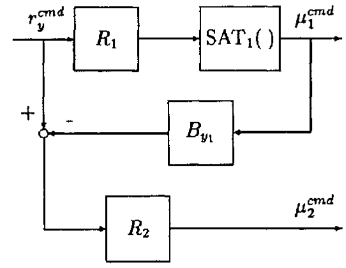

Fig. 5 Daisy chain control allocation.

The daisy chain allocation structure for redundant groups of controls is shown in Fig. 5. The daisy chain control allocation technique allocates redundant groups in a prioritized manner: elements of μ_2 are not used until at least one element of μ_1 is saturated. The daisy chain control allocation is given by the following equation:

$$
\rho_{\mathrm{dc}}\left(r_y^{\mathrm{cmd}}\right) = \begin{bmatrix} \mathrm{SAT}_1\left(R_1 r_y^{\mathrm{cmd}}\right) \\ R_2\left\{r_y^{\mathrm{cmd}} - B_{y_1}\mathrm{SAT}_1\left(R_1 r_y^{\mathrm{cmd}}\right)\right\} \end{bmatrix} = \begin{bmatrix} \mu_1^{\mathrm{cmd}} \\ \mu_2^{\mathrm{cmd}} \end{bmatrix} \tag{22}
$$

where SAT_1 is an $m_1 \times 1$ vector of standard saturation functions, m_1 is the number of elements in the first control group, and R_i is any right inverse of B_{y_i}, i.e., $B_{y_i} R_i = I$ for $i = 1, 2$.

According to Eq. (11), the attainable y-rate subset corresponding to the daisy chain technique is given by

$$
\Phi_{\mathrm{dc}}^{y} \equiv \left\{ r_y^{\mathrm{cmd}} \,\middle|\, B_y \rho_{\mathrm{dc}}\left(r_y^{\mathrm{cmd}}\right) = r_y^{\mathrm{cmd}},\; u^{\mathrm{cmd}} = \rho_{\mathrm{dc}}\left(r_y^{\mathrm{cmd}}\right) \in \Omega \right\} \subset \mathbb{R}^{n_y} \tag{23}
$$

where ρ_{dc} is given in Eq. (22). Although computation of Φ_{dc}^{y} is not critical for analyzing the stability of systems with control allocation, the concept of Φ_{dc}^{y} is important to understand the locality limitations of the analysis. It is beyond the scope of this paper to address computation of Φ_{dc}^{y}; however, knowledge of Φ_{dc}^{y} may be worthwhile for comparing the daisy chain to other allocation techniques.

With the daisy chain control allocation, the nonredundant system in Eq. (10) becomes

$$
\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_{zz} & A_{zy} \\ A_{yz} & A_{yy} \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} B_z \rho_{\mathrm{dc}}\left(r_y^{\mathrm{cmd}}\right) \\ r_y^{\mathrm{cmd}} \end{bmatrix} \tag{24}
$$

It is useful to investigate the stability of the zero dynamics due to the closed-loop stability implications mentioned in Sec. II. The zero dynamics with the daisy chain are derived by inserting the daisy chain allocation of Eq. (22) into the general zero dynamics expression of Eq. (14). Algebraic manipulations result in the following zero dynamics:

$$
\begin{aligned}
\dot{z} &= \hat{A}z + \hat{B}\,\mathrm{SAT}_1(\hat{C}z), &\qquad \hat{A} &\equiv A_{zz} - B_{z_2} R_2 A_{yz} \\
\hat{B} &\equiv B_{z_1} - B_{z_2} R_2 B_{y_1}, &\qquad \hat{C} &\equiv -R_1 A_{yz}
\end{aligned}
\tag{25}
$$

The following theorem states sufficient asymptotic stability conditions for the zero dynamics throughout the attainable y-rate subset defined by Eq. (23).

Theorem. The origin of the system in Eq. (25) is asymptotically stable if $\hat{A}$ is Hurwitz and

$$
\|\hat{C}(sI - \hat{A})^{-1}\|_\infty \|\hat{B}\| < 1
$$

Proof. Consider the Lyapunov function $V = z^T P z$, where $P > 0$. The time derivative of this function along trajectories of the dynamical system in Eq. (25) is given by

$$
\dot{V} = z^T(\hat{A}^T P + P\hat{A})z + 2z^T P\hat{B}\,\mathrm{SAT}_1(\hat{C}z) \tag{26}
$$

According to the theorem statement, a finite $\gamma > 0$ always exists such that

$$
\|\hat{C}(sI - \hat{A})^{-1}\|_\infty < \gamma < \|\hat{B}\|^{-1} \tag{27}
$$

and it follows from the bounded real lemma[18] that there exists a $P = P^T > 0$ such that

$$
\hat{A}^T P + P\hat{A} + \gamma^{-2} P P + \hat{C}^T\hat{C} < 0 \tag{28}
$$

Using the solution to the Riccati inequality in the Lyapunov function leads to another expression for the derivative of the Lyapunov function that follows by completing squares:

$$\dot{V} \leq -z^T \hat{C}^T \hat{C} z + \gamma^2 \mathrm{SAT}_1^T(\hat{C}z)\hat{B}^T \hat{B}\, \mathrm{SAT}_1(\hat{C}z) - \left[\gamma^{-1} z^T P - \gamma \mathrm{SAT}_1^T(\hat{C}z)\hat{B}^T\right]\left[\gamma^{-1} P z - \gamma \hat{B}\, \mathrm{SAT}_1(\hat{C}z)\right] \tag{29}$$

Using vector norm properties and noting that the last term of the preceding expression is negative semidefinite, the Lyapunov derivative is bounded by

$$\dot{V} \leq -\|\hat{C}z\|^2 + \gamma^2 \|\hat{B}\, \mathrm{SAT}_1(\hat{C}z)\|^2 \tag{30}$$

Because $\|\hat{B}\, \mathrm{SAT}_1(\hat{C}z)\| < \|\hat{B}\|\|\hat{C}z\|$, the following holds:

$$\dot{V} \leq [(\gamma\|\hat{B}\|)^2 - 1]\|\hat{C}z\|^2 \tag{31}$$

Recall that $\gamma < \|\hat{B}\|^{-1}$, which implies that $[(\gamma\|\hat{B}\|)^2 - 1] < 0$, and therefore $\dot{V} \leq 0$. Note that $\dot{V} < 0$ if $\|\hat{C}z\| > 0$, and so $\dot{V} = 0$ only if $\|\hat{C}z\| = 0$, which implies that $\dot{V} = 0$ only if $\hat{C}z = 0$. When $\hat{C}z = 0$, the z dynamics are $\dot{z} = \hat{A}z$ because $\mathrm{SAT}_1(0) = 0$. $\hat{A}$ Hurwitz implies that $z = 0$ is a globally asymptotically stable equilibrium point when $\dot{V} = 0$. Therefore, it is concluded by LaSalle's theorem[19] that the origin of the system in Eq. (14) is globally asymptotically stable. □

The following example demonstrates the results stated in the theorem. An aircraft example is given to show what type of systems are amenable to daisy chain control allocation and inversion control techniques.

Example 2

Consider state feedback augmentation of the longitudinal short-period dynamics of an aircraft with a constrained elevator (δ_e) and pitch thrust vectoring nozzle (δ_{tv}):

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & Z_\alpha \\ M_\alpha & M_q \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} & Z_{\delta_{tv}} \\ M_{\delta_e} & M_{\delta_{tv}} \end{bmatrix}\begin{bmatrix} \delta_e \\ \delta_{tv} \end{bmatrix}$$

$$\underline{\delta}_e \leq \delta_e \leq \bar{\delta}_e \qquad \underline{\delta}_{tv} \leq \delta_{tv} \leq \bar{\delta}_{tv} \tag{32}$$

The states are angle of attack α and body axis pitch rate q. Assume that a control command requirement exists that thrust vectoring is to be used only when the elevator saturates. Assume further that the actuators for both control effectors have bandwidths well beyond the short-period bandwidth. We now have a feedback control problem of the form in Fig. 1. Choose a controller of the form given in Eq. (12) with a daisy chain control allocator that satisfies the control command requirement. The daisy chain is defined by

$$\begin{bmatrix} \delta_e^{\mathrm{cmd}} \\ \delta_{tv}^{\mathrm{cmd}} \end{bmatrix} = \rho_{\mathrm{dc}}\left(r_q^{\mathrm{cmd}}\right) = \begin{bmatrix} \mathrm{sat}_e\left(\dfrac{r_q^{\mathrm{cmd}}}{M_{\delta_e}}\right) \\ \dfrac{1}{M_{\delta_{tv}}}\left[r_q^{\mathrm{cmd}} - M_{\delta_e}\mathrm{sat}_e\left(\dfrac{r_q^{\mathrm{cmd}}}{M_{\delta_e}}\right)\right] \end{bmatrix} \tag{33}$$

Combination of the open-loop plant and the daisy chain control allocator gives the following system for all $r_q^{\mathrm{cmd}} \in \Phi_{\mathrm{dc}}^q$:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} Z_{\delta_e} & Z_{\delta_{tv}} \end{bmatrix}\rho_{\mathrm{dc}}\left(r_q^{\mathrm{cmd}}\right) \\ r_q^{\mathrm{cmd}} \end{bmatrix}$$

$$= \begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_{\delta_e}\mathrm{sat}_e\left(\dfrac{r_q^{\mathrm{cmd}}}{M_{\delta_e}}\right) + \dfrac{Z_{\delta_{tv}}}{M_{\delta_{tv}}}\left[r_q^{\mathrm{cmd}} - M_{\delta_e}\mathrm{sat}_e\left(\dfrac{r_q^{\mathrm{cmd}}}{M_{\delta_e}}\right)\right] \\ r_q^{\mathrm{cmd}} \end{bmatrix} \tag{34}$$

The theorem states that the zero dynamics with respect to input r_q^{cmd} and output q are asymptotically stable for all $r_q^{\mathrm{cmd}} \in \Phi_{\mathrm{dc}}^q$ if the following conditions hold.

Condition 1. $Z_\alpha - \left(Z_{\delta_{tv}}/M_{\delta_{tv}}M_\alpha\right)$ is Hurwitz.

Condition 2.

$$\left\|M_\alpha\left[sI - Z_\alpha + \left(Z_{\delta_{tv}}/M_{\delta_{tv}}\right)M_\alpha\right]^{-1}\right\|_\infty \times \left\|\left(Z_{\delta_e}/M_{\delta_e}\right) - \left(Z_{\delta_{tv}}/M_{\delta_{tv}}\right)\right\| < 1$$

Note these conditions are satisfied if $Z_\alpha - (Z_{\delta_{tv}}/M_{\delta_{tv}})M_\alpha < 0$ and if $(Z_{\delta_e}/M_{\delta_e}) - (Z_{\delta_{tv}}/M_{\delta_{tv}})$ is sufficiently small. If the elevator and thrust vectoring nozzle provide sufficiently more normalized moment than normalized direct lift, then the control ratios will be sufficiently small to satisfy condition 2 provided that condition 1 is satisfied. For a nondamaged conventional aircraft, typically $Z_\alpha < 0$ and $Z_{\delta_{tv}}/M_{\delta_{tv}}$ is sufficiently small to satisfy condition 1. Because δ_e and δ_{tv} are well aft of the center of gravity, $Z_{\delta_{tv}}/M_{\delta_{tv}}$ and $Z_{\delta_e}/M_{\delta_e}$ are sufficiently small to satisfy condition 2 as indicated by the following data.

Data for an F-18 supermaneuverable aircraft[2] at Mach 0.6, 30,000-ft altitude, and 5.2-deg angle of attack are given by the following:

$$\begin{bmatrix} Z_\alpha & Z_q \\ M_\alpha & M_q \end{bmatrix} = \begin{bmatrix} -0.5088 & 0.994 \\ -1.131 & -0.2804 \end{bmatrix}$$

$$\begin{bmatrix} Z_{\delta_e} & Z_{\delta_{tv}} \\ M_{\delta_e} & M_{\delta_{tv}} \end{bmatrix} = \begin{bmatrix} -0.09277 & -0.01787 \\ -6.573 & -1.525 \end{bmatrix}$$

$$-24 \leq \delta_e \leq 10.5 \qquad -30 \leq \delta_{tv} \leq 30 \tag{35}$$

The controls are in degrees, α is in degrees, and q is in degrees per second. These data result in the following quantities that satisfy the theorem conditions:

$$Z_\alpha - \left(Z_{\delta_{tv}}/M_{\delta_{tv}}\right)M_\alpha = -0.4935 < 0 \Rightarrow \text{Hurwitz}$$

$$\left\|M_\alpha\left[sI - Z_\alpha + \left(Z_{\delta_{tv}}/M_{\delta_{tv}}\right)M_\alpha\right]^{-1}\right\|_\infty \times \left\|\left(Z_{\delta_e}/M_{\delta_e}\right) - \left(Z_{\delta_{tv}}/M_{\delta_{tv}}\right)\right\| = 0.0064 < 1 \tag{36}$$

Note that the individual controls have a destabilizing tendency on the zero dynamics as indicated by the increase in effective Z_α due to the controls:

$$Z_\alpha = -0.5088 < Z_\alpha - (Z_\delta/M_\delta)\, M_\alpha \tag{37}$$

where δ represents the elevator or the thrust vector angle. However, Z_α dominates; thus the zero dynamics are asymptotically stable.

Recall that satisfaction of the theorem requirements is interpreted as meaning that a daisy chain control allocation will not introduce unstable zero dynamics into the nonredundant system for all $r_q^{\mathrm{cmd}} \in \Phi_{\mathrm{dc}}^q$. For this simple example, the q-rate subset is easily computed to be

$$\Phi_{\mathrm{dc}}^q = \left\{r_q^{\mathrm{cmd}} \,\middle|\, -114.8 \leq r_q^{\mathrm{cmd}} \leq 203.5\right\} \tag{38}$$

Thus, it can be ensured that a dynamic inversion controller of the form given in Eq. (12) along with a daisy chain allocator will asymptotically stabilize the original redundant system for all $r_q^{\mathrm{cmd}} \in \Phi_{\mathrm{dc}}^q$.

In example 2, the theorem reaffirms well-known knowledge that pitch rate zeros are generally stable for nondamaged, conventional aircraft with surfaces aft of the center of gravity. Also the single output and single input per control group nature of example 2 allows stability analysis in a sequential manner and does not necessarily warrant use of the current result. Generally, however, for multiple-input, multiple-output systems, this sequential analysis is not valid since all control group elements do not saturate simultaneously. In the following flight control example for a nonconventional aircraft, multiple inputs affect the lateral and directional axes of a tailless aircraft, thus complicating the control allocation problem.

Example 3

Consider the following linear approximation of the lateral/directional dynamics of an aircraft without a vertical tail.[20] The data are at Mach 0.4, 15,000-ft altitude, and 8.8-deg angle of attack:

$$\begin{bmatrix} \dot\beta \\ \dot p \\ \dot r \end{bmatrix} = \begin{bmatrix} -0.004 & 0.154 & -0.988 \\ -8.21 & -0.785 & 0.117 \\ -0.889 & -0.030 & -0.016 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \end{bmatrix} + \begin{bmatrix} -0.009 & 0.013 & 0.006 & 0.011 & -0.004 \\ 7.57 & -4.97 & 3.65 & 0.079 & -0.614 \\ 0.091 & -0.181 & -0.416 & -0.804 & 0.188 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_s \\ \delta_t \\ \delta_{\mathrm{ytv}} \\ \delta_f \end{bmatrix} \tag{39}$$

The states are sideslip angle β, body-axis roll rate p, and body-axis yaw rate r; the angles are expressed in degrees and angular rates in degrees per second. The constrained controls are differential elevons δ_e, differential spoiler/slot-deflectors δ_s, differential all moving tips δ_t, yaw thrust vectoring δ_{ytv}, and differential outboard leading-edge flaps δ_f, which are all expressed in degrees. It is desired to command the roll and yaw rates, which leads to the following state partition in terms of the general system in Eq. (1):

$$z = \beta, \qquad y = [\,p \quad r\,]^T \tag{40}$$

Assume the following naive control group partition for no particular reason:

$$\left[\begin{array}{c} \mu_1 \\ \hline \mu_2 \end{array}\right] = \left[\begin{array}{c} \delta_e \\ \delta_s \\ \hline \delta_t \\ \delta_{\mathrm{ytv}} \\ \delta_f \end{array}\right] \tag{41}$$

The state and control partitions lead to the following matrices corresponding to the general system:

$$A_{zz} = -0.004, \qquad A_{yz} = \begin{bmatrix} -8.21 \\ -0.889 \end{bmatrix}$$

$$B_{z_1} = [\,-0.009 \;\; 0.013\,], \qquad B_{z_2} = [\,0.006 \;\; 0.011 \;\; -0.004\,]$$

$$B_{y_1} = \begin{bmatrix} 7.57 & -4.97 \\ 0.091 & -0.181 \end{bmatrix}$$

$$B_{y_2} = \begin{bmatrix} 3.65 & 0.079 & -0.614 \\ -0.416 & -0.804 & 0.188 \end{bmatrix} \tag{42}$$

The inverse of B_{y_1} is used for the first control group in the daisy chain structure since it is unique. The pseudoinverse will be used for the second control group:

$$R_1 = B_{y_1}^{-1}, \qquad R_2 = B_{y_2}^T \left(B_{y_2} B_{y_2}^T\right)^{-1} \tag{43}$$

These data result in the following quantities that do not satisfy the theorem conditions:

$$\hat A = -0.0149 < 0 \Rightarrow \text{Hurwitz}$$

$$\left\| \hat C \left(sI - \hat A\right)^{-1} \right\|_\infty \|\hat B\| = 7.1 > 1 \tag{44}$$

Thus, a daisy chain control allocator may introduce unstable zero dynamics into the system and thus may destabilize a closed-loop system with a dynamic inversion controller.

As seen in example 3, a damaged or tailless aircraft may have significantly reduced A_{zz} (Z_α or Y_β), and this reduced A_{zz} along with additional nonconventional effectors may destabilize the rate zeros. Although not specifically addressed in the examples, phugoid and spiral characteristics may also be adversely affected by improper control allocation.

IV. Conclusions

It has been shown that with the concept of control allocation, feedback control of a redundant system can be reduced to feedback control of a nonredundant system. This permits control command requirements to be addressed separate from feedback control design, while possibly reducing design complexity. It has also been shown that control allocation can introduce unstable zero dynamics into the nonredundant system that may destabilize certain closed loops. The daisy chain control allocation technique is presented for redundant groups of controls, and sufficient conditions are derived that ensure the daisy chain control allocation technique does not introduce unstable zero dynamics into the nonredundant system. The utility of the derived result has been demonstrated with flight control examples.

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